Abstract. While it is usually assumed that cooperative agents may partition themselves in an arbitrary coalition structure, one can think of many physical and organizational constrains that limit the number of co-existing coalitions. In this paper we introduce \( k \)-coalitional cooperative games – a subclass of partition function games – especially designed to model such situations. For the new games we propose a dedicated extension of the Shapley value and study its computational properties. In particular, we show that, under some conditions, it can be computed in polynomial time given two existing representations for coalitional games with externalities.

Keywords: Cooperative games, Shapley value, externalities

1 Introduction

Many works in both game theory and computer science consider coalitional games in their general, unrestricted form, where, among others, it is assumed that all cooperative arrangements are feasible [7, 2]. In particular, in characteristic function games, the players involved are allowed to partition themselves into any set of coalitions, called a coalition structure. Moreover, in this model, it is assumed though that no coalition influences the value of a co-existing coalition, \( i.e. \), the value of a coalition is independent from the coalition structure. In more general partition function games [20], players are also allowed to form any coalition structure, but now the value of any coalition may depend on other coalitions.

Despite popularity of the above general models, for long, it has been recognized that many potential applications of coalitional games involve various forms of restrictions. Perhaps the most prominent example are graph-restricted games due to Myerson [11], where it is assumed that cooperation is only feasible in coalitions, in which agents are either directly and indirectly connected in an underlying graph. Myerson’s games were studied by various authors in Artificial Intelligence and Multi-Agent Systems [9, 17]. An interesting alternative to the Myerson model was proposed by See et al. [14] who studied weighted voting games restricted by a graph representing potential alliances between parties. Here, only coalitions of agents that form a clique in the underlying graph can cooperate. Yet another model was proposed by Rahwan et al. [13], where restrictions on feasible coalitions are expressed in the form of logical constraints. Furthermore, Meir et al. [9] consider games with coalitions of restricted size.

The common feature of all the above models is that they focus on restricting feasible coalitions. As a result, coalition structures are restricted only indirectly, \( i.e. \), by the fact...
that they cannot contain infeasible coalitions. However, one can think of many physical and organizational restrictions that place constraints on feasible coalition structures and not coalitions. During the Cold War any country in Europe belonged either to NATO, the Warsaw Pact, or remained neutral. Thus, if we consider all neutral countries as a coalition, then any feasible coalition structure would have at most three coalitions. Also the number of parliamentary factions is in many countries, like Japan or Germany, limited either by rules or tradition. Similar restrictions are likely to occur in a multi-agent system, where, due to cost considerations, only a few agents may be sophisticated enough to play the role of coalition coordinators/leaders (see, e.g., the work by Coviello and Franceschetti [3] for an analysis of the problem of assigning followers to leaders in such a case). Thus, any coalition is feasible but no coalition structure may contain more coalitions than they are leaders in the system.

Against this background, in this paper we introduce $k$-coalitional cooperative games – a subclass of partition function games especially designed to model the above situations. In particular, we assume that any coalition may form but the number of coalitions in any coalition structure is limited by constant $k$. For the new games, we propose a dedicated extension of the Shapley value [15] – a fundamental solution concepts for coalitional games. We prove that the new value is uniquely defined by a set of axioms that closely follow Shapley’s original axiomatization. Furthermore, we analyse the computational properties of the new value and show that it can be computed in polynomial time, if the $k$-coalitional game is represented with Partition Decision Trees [18].

2 Preliminary Definitions

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents with $|N| = n$. A coalition, $S$, is any non-empty subset of $N$. A game without externalities (in the characteristic function form) is given by a function $\hat{v}$ that associates a real number with every coalition of agents: $\hat{v} : 2^N \to \mathbb{R}$ with the assumption $\hat{v}(\emptyset) = 0$. Formally, game is a pair $(N, \hat{v})$, but we refer to it by $\hat{v}$ only. We denote the set of all games without externalities by $\mathcal{CG}$.

In a more general model with externalities, the value of a coalition depends on the members of a coalition, but also on co-existing coalitions. A partition of $N$ (also known as a coalition structure) is a set of disjoint coalitions that collectively cover $N$. A pair $(S,P)$, where $P$ is a partition and $S \in P$, is called an embedded coalition. The set of all partitions is denoted by $\mathcal{P}$, and all embedded coalitions – by $\mathcal{EC}$. Now, in a game with externalities (in the partition function form) $(N,v)$, the function $v$ associates a real number with every embedded coalition in every partition, i.e., $v : \mathcal{EC} \to \mathbb{R}$ (with $v(\emptyset, P) = 0$ for every partition $P \in \mathcal{P}$). We denote the set of all games with externalities by $\mathcal{PG}$.

Our notation: We use a shorthand notation for set subtraction and set union operations: $N \setminus S = N \setminus S$ and $S + \{i\} = S \cup \{i\}$. Often, we omit brackets and simply write $S + i$. To denote the partition obtained by the transfer of agent $i$ to coalition $T$ in partition $P$, we introduce the following notation:

$$\tau^T_i(P) = P \setminus \{P(i), T\} \cup \{P(i)_{-i}, T_{+i}\},$$
where $P(i)$ denotes $i$'s coalition in $P$. For function $f : N \rightarrow X$ and subset $S \subseteq N$ we define $f(S) = \{f(i) \mid i \in S\}$. and, in the same manner, for a set of sets $P = \{S_1, \ldots, S_m\}$, we have $f(S) = \{f(S_1), \ldots, f(S_m)\}$. Also, $f^{-1} : X \rightarrow 2^N$ is an inverse function, i.e., $f^{-1}(x) = \{i \in N \mid f(i) = x\}$ for $x \in X$. In particular, combining above definitions, $f^{-1}(X)$ forms a partition of $N$.

**The Shapley value:** Assume that the grand coalition, i.e., coalition of all agents, forms. Now, a value of a game is a vector that distributes the payoff of the grand coalition among the agents. The value of player $i$ in game $v$ is denoted $\varphi_i(v)$. In his seminal work, Shapley [15] proved that there exists a unique division scheme in games without externalities that satisfies the following four axioms:

- **Efficiency** (the entire payoff is distributed among agents): $\sum_{i \in N} \varphi_i(\hat{v}) = \hat{v}(N)$ for every $\hat{v}$;
- **Symmetry** (payoffs do not depend on the agents’ names): $\varphi_i(f(\hat{v})) = \varphi(f(\hat{v}))$ for every $\hat{v}$ and every $f : N \rightarrow N$;
- **Additivity** (the sum of payoffs in two separate games equals the payoff in the combined game): $\varphi(\hat{v}_1 + \hat{v}_2) = \varphi(\hat{v}_1) + \varphi(\hat{v}_2)$ for all $\hat{v}_1, \hat{v}_2$;
- **Null-player Axiom** (agents with no contribution to any coalition should get nothing): if $\hat{v}(S) = \hat{v}(S \setminus \{i\})$ for every $S \subseteq N, i \in S$, then $\varphi_i(\hat{v}) = 0$, for every $\hat{v}$;

where games $f(\hat{v})$ (for bijection $f$) and $\hat{v}_1 + \hat{v}_2$ are defined as follows: $f(\hat{v})(S) = \hat{v}(f(S))$, and $(\hat{v}_1 + \hat{v}_2)(S) = \hat{v}_1(S) + \hat{v}_2(S)$. This unique solution is known as the Shapley value:

$$SV_i(\hat{v}) = \sum_{S \subseteq N \atop i \in S} \frac{(|S| - 1)!([N] - |S|)!}{|N|!} (\hat{v}(S) - \hat{v}(S \setminus \{i\})). \quad (1)$$

As an intuition, Shapley provided, the following process that leads to his value. Assume that agents enter the game in a random order with an aim to form the grand coalition. As agent $i$ enters, he receives a payoff that equals his marginal contribution to the group of agents that he joins: $[mc_i(v)](S) = \hat{v}(S) - \hat{v}(S \setminus \{i\})$. Now, the Shapley value is the expected outcome of agents’ contributions over all orders.

To formalize this description, let us introduce some additional notation. A set of all permutations (orders) of $N$ is denoted by $\Pi$. For a given permutation $\pi$, the set of agents that appear in $\pi$ before $i$ is denoted $A_{i+}^\pi$, and after $-Z_{i+}^\pi$. If we include $i$ in this sets we write $A_{i+}^\pi$ and $Z_{i+}^\pi$. Now, the Shapley value is alternately defined by the following formula:

$$SV_i(\hat{v}) = \frac{1}{|N|!} \sum_{\pi \in \Pi} \hat{v}(A_{i+}^\pi) - \hat{v}(A_i^\pi). \quad (2)$$

**Extended Shapley values:** Consider the translation of Shapley’s axioms to games with externalities. While Efficiency, Symmetry and Additivity can be easily adapted, the Null-player Axiom poses a problem – how should the contribution of an agent to an embedded coalition be defined? In games without externalities this is a difference between the value of a coalition with and without a player. But in games with externalities,
when agent leaves coalition $S \cup \{i\}$ in partition \(\{S \cup \{i\}, T_1, \ldots, T_k\}\), the effect of his move depends on what other coalition he joins, as the value of $S$ when $i$ joins $T_1$ may differ than the one when $i$ joins $T_2$. A change associated with a transfer of agent $i$ from coalition $S \cup \{i\}$ that result in partition $P$ is denoted by

\[
[emc_i(v)](S, P) = v(S + i, \tau_i^S(P)) - v(S, P),
\]

and called the \textit{elementary marginal contribution}.

In the most strict definition of a null-player, we assume that every transfer does not change the value of a coalition, i.e., every elementary marginal contributions has value zero. Unfortunately, the Null-player Axiom based on this definition combined with the three other axioms (the standard translation) is too weak to imply uniqueness. To overcome this problem, a number of researchers proposed to define the marginal contribution of an agent as a weighted average over elementary marginal contributions associated with leaving a given embedded coalition \([1, 12, 4, 5, 16]\). Specifically, the marginal contribution of agent $i$ to coalition $S$ in partition $P$ is defined as follows:

\[
[mc_i(v)](S, P) = \sum_{T \in P \setminus S \cup \{\emptyset\}} \alpha_i(S - i, \tau_i^T(P))[emc_i(v)](S - i, \tau_i^T(P)),
\]

for some weights $\alpha$. Here, empty set in the sum corresponds to creating of a new coalition. Now, agent $i$ is a $\alpha$-null-player if all his marginal contributions equal zero. Skibski et al. [19] provided two general results about this approach – firstly, for every $\alpha$, Efficiency, Symmetry, Additivity and the $\alpha$-Null-player Axiom is sufficient to imply uniqueness; secondly, every value that satisfies the standard translation of Shapley’s axioms can be obtained with this approach for some weights $\alpha$. The reader is referred to the work by Skibski et al. [19] for a recent overview of the topic of the Shapley value in games with externalities.

3 The $k$-Coalitional Games and the Shapley Value

In this section, we introduce the new class of coalitional games with partitions of a bounded size and propose a dedicated extension of the Shapley value.

Define $K = \{1, 2, \ldots, k\}$. A size of the partition, denoted by $|P|$, is the number of coalitions it consists of. We denote the set of all partitions of size $m$ by $\mathcal{P}_m$. Now, a $k$-coalitional game is a pair $(N, v)$, where $v$ is a function $v : \bigcup_{m \in K} \mathcal{P}_m \to \mathbb{R}$ that associates a real number with every coalition in a partition of size at most $k$, where $k \leq |N|$. We denote the set of all $k$-coalitional games by $\mathcal{PG}_k$.

Now, we introduce a dedicated solution concept for $k$-coalitional games. We start by presenting the process, similar to the one proposed by Shapley [15], which yields the value in expectation. To this end, assume that agents leave (instead of enter) the grand coalition in a random order through one of the $k-1$ exits/doors. We assume that agents that left through a given exit form a coalition; thus, all agents are partitioned according to their selected exits. As agent $i$ leaves, he chooses each exit with the same probability \(\frac{1}{k-1}\) and receives a payoff that equals his (elementary) marginal contribution to the group of agents that he left. More formally, assuming the agents in $S$ have not yet
left, other agents are partitioned into \( P \setminus S \), and agent \( i \) chose the same exit as coalition \( T \), the elementary marginal contribution equals \( v(S, P) - v(S - i, T(i)) \). Now, the \( k \)-coalitional Shapley value is the expected outcome of the agent’s elementary marginal contribution over all possible orders.

Formally, the \( k \)-coalitional Shapley value is a function \( \varphi^k : \mathcal{P}^k \rightarrow \mathbb{R}^N \) defined with the following formula:

\[
\varphi^k_i(v) = \sum_{\pi \in \Pi} \sum_{f : A^k_\pi \rightarrow K - 1} \left[ \text{emc}_i(v)(S, P) \right] \left[ Z_{i+}^f, Z_{i-}^f \cup f^{-1}(K - 1) \right] / |N|!(k - 1)^{|A^k_\pi|}.
\] (5)

We start the analysis of the value by deriving a more concise version of the above formula.

**Lemma 1.** The \( k \)-coalitional Shapley value satisfies:

\[
\varphi^k_i(v) = \sum_{(S, P) \in EC, \forall q \notin S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} p(P \setminus S)[\text{emc}_i(v)](S, P),
\] (6)

where \( p(P \setminus S) = \frac{(k - 1)!}{(k - |P|)!(k - 1)^{|N|}}}.

**Proof.** Let us calculate the probability of a given transfer corresponding to elementary marginal contribution \( [\text{emc}_i(v)](S, P) \). Firstly, in a permutation, \( i \) has to be exactly before agents from \( S \) and after agents from \( N \setminus (S \cup \{i\}) \), which happens with the probability \( \frac{|S|!(|N| - |S| - 1)!}{|N|!} \). Secondly, the exit of agent \( i \) should lead to partition \( P \). Thus, first (in a permutation) agents from each coalition in \( P \setminus S \) can choose arbitrary – but different – exits (they have exactly \( (k - 1) \cdot \cdots \cdot (k - |P| + 1) \) such choices), and then each of the remaining agents has to choose exactly the same one.

Let us formalize the notion of the marginal contribution in our value. Assume that \( i \in S \in P \). Based on the process described above, the probability that agent \( i \) will join any coalition \( T \in P \) equals \( 1/(k - 1) \). Thus, the chance of creating a new coalition equals \( (k - |P|)/(k - 1) \):

\[
\alpha^k_i = \frac{k - |P|}{k - 1}, \quad \text{if } P(i) = \{i\}, \quad \alpha^k_i = \frac{1}{k - 1}, \quad \text{otherwise}.
\]

and the marginal contribution follows formula (4) with weights \( \alpha^k \). The corresponding \( \alpha^k \)-Null-player Axiom states that a player who has zero marginal contribution \( [\text{mc}_i^k(v)](S, P) \) (formula (4) with weights \( \alpha^k \)) has zero payoff. Now, we can introduce the axiomatization of the value.

**Theorem 1.** The \( k \)-coalitional Shapley value is the only value that satisfies Efficiency, Symmetry, Additivity, and the \( \alpha^k \)-Null-player Axiom.

**Proof.** First, we show that the \( k \)-coalitional Shapley value satisfies all four axioms. To argue that Efficiency is satisfied, consider formula (5). Let us fix permutation \( \pi \in \Pi \) and function \( f \). Considering the sum of elementary marginal contributions:
\[ \sum_{i \in N} [emc_i(v)(Z^i_T, Z^i_T \cup f^{-1}(K_{-1}))] \text{ we see that it sums up to } v(N, \{N, \emptyset\}) \text{ (as the grand coalition, } N, \text{ dissolves sequentially to the empty coalition, } \emptyset, \text{ with zero value).} \]

The fact, that the value satisfies the \( \alpha^k \)-Null-player Axiom follows from Lemma 1. To see that, note that if \( i \notin S \), then \( v(S, P) \) appears in the formula only once. On the other hand, if \( i \in S \), then \( v(S, P) \) appears once for each possible transfers of \( i \) from \( S \) outside with weight \((|S| - 1)!(|N| - |S|)!/|N|! \text{ times } (k - 1)!/(k - |P|)!(k - 1)!^{N - |S| + 1})\) for transfers to existing coalitions, and with weight \( k - |P| \) replaced by \( k - |P| + 1 \) for a transfer to a new coalition. The proper reformulation leads to the following formula:

\[
\varphi^k_i(v) = \sum_{(S, P) \in EC, i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \frac{(|S| - 1)}{p(P \setminus S)}[mc^k_i(v)](S, P),
\]

which proves that if all marginal contributions of player \( i \) equals zero, then he gets zero payoff.

The proof of uniqueness follows from [19, Theorem 1].

Before proceeding, let us show that all four axioms are required to guarantee uniqueness:

- \( \varphi_i(v) = 0 \) satisfies Symmetry, Additivity, and the \( \alpha^k \)-Null-player Axiom, but not Efficiency;
- for fixed \( \pi \) (e.g., \( \pi = (1, 2, \ldots, n) \)),
\[
\varphi^k_i(v) = \sum_{f: A^*_i \to K_{-1}} \frac{[emc_i(v)](Z^i_T, Z^i_T \cup f^{-1}(K_{-1}))}{|N|!(k - 1)|A^*_i|}
\]

satisfies Efficiency, Additivity, and the \( \alpha^k \)-Null-player Axiom, but not Symmetry;
- \( \varphi_i(v) = v(N, \{N\})/(|N| - |D|) \) for \( i \notin D \), and \( \varphi_i(v) = 0 \) otherwise, where \( D \) is a set of \( \alpha^k \)-null-players satisfies Efficiency, Symmetry, and the \( \alpha^k \)-Null-player Axiom, but not Additivity;
- \( \varphi_i(v) = v(N, \{N\})/|N| \) satisfies Efficiency, Symmetry, and Additivity, but not the \( \alpha^k \)-Null-player Axiom.

From all extensions of the Shapley value to games with externalities only one – the McQuillin value [8] – can be applied to our restricted setting. Let us define the game without externalities as follows: \( \hat{\nu}^{M\nu Q}(S) = v(S, \{S, N \setminus S\}) \). The McQuillin value is the Shapley value for this game:

\[
\varphi^{M\nu Q}_i(v) = SV_i(\hat{\nu}^{M\nu Q}).
\]

We show below that the McQuillin’s value is a special case of the \( k \)-coalitional Shapley value.

**Proposition 1.** The McQuillin value is the 2-coalitional Shapley value.

**Proof.** Consider the \( k \)-coalitional value with \( k = 2 \). As we recall the interpretation of our process, we see there is only one exit and all agents leaving coalition \( S \) form one group – \( N \setminus S \). Thus, the marginal contribution for every permutation coincides with marginal contribution in game \( \hat{\nu}^{M\nu Q} \); hence, both values are equal. \( \square \)
As already mentioned, other extensions of the Shapley value to games with externalities cannot be directly applied to our environment. However, one can consider the following indirect application. Let \( v^* \) be a partition-function form game created from \( v \in PG_k \) by assigning zero values to all coalitions embedded in partitions with more than \( k \) coalitions: \( v^*(S, P) = v(S, P) \) if \( |P| \leq k \) and \( v^*(S, P) = 0 \), otherwise. Now, technically, every value for partition-function form games can be applied to \( v^* \).

While at first such an indirect construction seems appealing, we note that it does not maintain some properties of \( k \)-coalitional game. For instance, player who normally does not affect any coalition value (with all elementary marginal contributions zero) will affect it now by forming \( k+1 \)th coalition. Thus, existing solutions for games with externalities may assign non-zero payoff to this player.

### 4 Computations Under Specific Representations

In this section, we discuss the complexity of calculating the \( k \)-coalitional Shapley value based on two representations proposed for games with externalities – Embedded MC-Nets [10] and Partition Decision Trees [18]. As we will prove, there exists an interesting connection between computing the \( k \)-coalitional Shapley value and the problem of \( k \)-colorings in a graph.

We start with a property of the \( k \)-coalitional Shapley value that is crucial from the computational perspective.

**Lemma 2.** The \( k \)-coalitional Shapley value satisfies the Strong Null-player Axiom: if agent is a null-player (in a strict sense) then he has no impact on the payoffs of others, i.e., he can be removed from the game.

**Proof.** Let agent \( j \) be a null-player. An elementary game \( e(S, P) \) is a game in which only coalition \((S, P)\) has non-zero value (equal one): \( e(S, P) = 1 \) if \((S', P') = (S, P)\), and \( e(S, P)(S', P') = 0 \), otherwise. For an embedded coalition \((S, P)\) such that \( j \in S \), consider game \( \tilde{v}(S, P) \) defined as:

\[
\tilde{v}(S, P) = e(S, P) + \sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} e(S - j, \tau_j^T(P)).
\]

We can easily check that agent \( j \) is a null-player in \( \tilde{v}(S, P) \), and collection of games \( \langle \tilde{v}(S, P) \rangle_{(S, P) \in EC,j \in S} \) forms a basis of class of games in which agent \( j \) is a null-player. Now, game \( \tilde{v}(S, P) \) with agent \( j \) removed simplifies to the elementary game \( e(S - j, P - j) \) where only coalition \((S - j, P - j)\) has non-zero value, which equals 1 (\( P - j \) is a partition of players over \( N \setminus \{j\} \)). Thus, it is enough to show that \( \varphi_i(N, \tilde{v}(S, P)) = \varphi_i(N - j, e(S - j, P - j)) \) holds for every \( i \in N \setminus \{j\} \).

Assume that \( i \not\in S \). From formula (6) we have:

\[
\varphi_i^k(\tilde{v}(S, P)) = \frac{|S|!(|N| - |S| - 1)!}{|N|!} p(P \setminus S) + \sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} p(\tau_j^T(P) \setminus S - j).\]
Recall that $p(P \setminus S) = \frac{(k-1)!}{(k-|P|)! \cdot (k-1)! \cdot |P|}$.

Simple calculations gives

$$
\sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} p(T \setminus (P \setminus S)) = \frac{(k-1)!}{(k-|P|)! \cdot (k-1)! \cdot |P|} \cdot \frac{(|P| - 1)!}{|N|} = p(P \setminus S),
$$

which equals $p(P \setminus S)$. Thus,

$$
\phi_k^i(E(S,P)) = p(P \setminus S) \left( \frac{(|P| - 1)!}{|N|} \cdot \frac{(|N| - |S|)!}{|N|} + \frac{(|N| - |S| - 1)!}{|N|} \right)
$$

As payoffs of all agents not from $S$ remain the same after removing player $j$, then based on efficiency and symmetry payoffs of agents from $S$ also do not change. $\square$

The importance of this lemma comes from the fact that most representations (the aim of which is to provide a concise description of the game) usually focus on modelling relationships between subsets of agents. Following Lemma 2, when calculating the value, we can limit ourselves to agents that matters for a given relationship.

The first representation that we discuss, the Embedded MC-Nets introduced by Michalak et al. [10], is an extension of the MC-Nets representation [6]. The basic building block of the original MC-Nets is a boolean expression over $N$ of the form:

$$
p_1 \land p_2 \land \ldots \land p_k \land \neg n_1 \land \neg n_2 \land \ldots \land \neg n_l,
$$

with $p_1, p_2, \ldots, p_k \in N$ being positive literals, and $n_1, n_2, \ldots, n_l \in N$ being negative literals. Coalition $S$ satisfies a given boolean expression ($\alpha$ or $\beta_i$) if it contains all agents from the positive literals, and does not contain any agent from the negative literals.

Now, one rule of Embedded MC-Nets is of the form:

$$
\alpha \mid \beta_1, \beta_2, \ldots, \beta_m \rightarrow w,
$$

where $w \in \mathbb{R}$, and $\alpha, \beta_1, \beta_2, \ldots, \beta_m$ are the standard MC-Nets boolean expressions. Embedded coalition $(S,P)$ satisfies the entire rule if $S$ satisfies $\alpha$ and, for every $\beta_i$, there exists a coalition $T \in P$ such that $T$ satisfies $\beta_i$. We assume also that agents that appear in a negative part of the rules appear also somewhere in the positive ones.

**Theorem 2.** Calculating the $k$-coalitional Shapley value from a single Embedded MC-Nets rule is equivalent to the problem of counting $(k-1)$-colorings of the graph. Thus, it is $\#P$-complete.

**Proof.** First, we assume that: $\alpha$ does not contain negative literals (each can be added as a separate boolean expression on the right-hand side), sets of positive literals in all boolean expressions $\alpha, \beta_1, \ldots, \beta_m$ do not overlap (if they do, they can be combined, or if not – for example if one of these expressions is $\alpha$ – the rule is contrary), and that each expression is not contrary. Now, based on the Strong Null-player Axiom (Lemma 2), we assume that the game only consists of agents that appear in the rule – $N$. Thus, the only coalition with a non-zero value in the game described using this rule is a coalition formed by agents from positive literals from $\alpha$. We will denote it by $S$. The value,
Fig. 1. Computing the $k$-coalitional Shapley value based on an Embedded MC-Nets rule using $k$-coloring. Here, $g(k)$ is a chromatic polynomial

However, is non-zero only in partitions that satisfy boolean expressions from the right-hand side. Thus, our goal is to compute the sum of weights of all such partitions. Recall that the weight of a partition is the number of functions $f : P \setminus S \rightarrow K_{-1}$ such that $f^{-1}(K_{-1}) = P \setminus S$ divided by $(k-1)^{|N|-|S|}$ (and using the process interpretation, the number of ways agents can leave the grand coalition and form partition $P \setminus S$ outside using $k$ exits).

To this end, consider a graph with $m$ nodes that corresponds to boolean expressions $\beta_1, \ldots, \beta_m$. Now, two expressions are connected if there exists a negative literal in one with agent from the other one. Thus, an edge means that these expressions cannot be merged. Now, each $(k-1)$-coloring of this graph (assigning colors from set $\{1, 2, \ldots, k-1\}$ in such a way that connected nodes have different colors) corresponds to one partition of agents that meets the boolean expressions, i.e., one function $f$ that meets the above criteria and the number of colorings divided by $(k-1)^{|N|-|S|}$ is equal to the sum of weights of partitions.

Example 1. First, we show how to obtain needed form of an Embedded MC-Nets formula. Consider formula $1 \land 2 \land \neg 5 \mid (3 \land \neg 5)(4 \land \neg 6)(5 \land \neg 7)(7 \land \neg 4)$. To eliminate negative literal $\neg 5$ prior to the bar we add (5) to the right-hand side: $1 \land 2 \mid (3 \land \neg 5)(4 \land \neg 6)(5 \land \neg 7)(7 \land \neg 4)(5)$. Now, we see that positive literals in two expressions overlap. Thus, we can combine them: $1 \land 2 \mid (3 \land \neg 5)(4 \land \neg 6)(5 \land \neg 7)(7 \land \neg 4)$. Now, the rest of our reasoning – a graph corresponding to this formula, its chromatic polynomial and the resulting value of player 1 – is presented on Figure 1.

Although the above result shows that computing our value is hard in general, the relation to $k$-coloring shows that whenever rules can be modelled by a graph for which counting the number of $k$-colorings is simple, we have an polynomial algorithm for our problem. We state two following corollaries.

Corollary 1. The 2-coalitional and 3-coalitional Shapley values can be calculated in polynomial time under the Embedded MC-Nets representation.
This result comes from trivial algorithms for problem of counting $k$-colorings for $k = 1, 2$. Another result comes from the restriction of the Embedded MC-Nets.

**Corollary 2.** The $k$-coalitional Shapley value can be calculated in polynomial time under the Embedded MC-Nets representation restricted to rules without negative literals.

Note that without negative literals the graph that represents restrictions in connecting formulas does not have edges. Another example of the application of Theorem 2 is Theorem 3.

With Partition Decision Trees [18], a game is represented as a set of rooted directed trees, called a **PDT rule**. Each PDT rule is a tree $(V, E)$, with root $x$ and two label functions $f_V$ and $f_E$ – non-leaf nodes are labelled with agents’ names, leaf nodes are labelled with payoff vectors, and edges are labelled with numbers that correspond to coalitions. Thus, one path defines a partition of agents and their value. Conciseness of this representation comes from two features. Firstly, not all agents have to appear on a path. Secondly, trees are additive (the value of embedded coalition is the sum of values of this coalition in every tree), thus separate concise rules can be independently described.

**Theorem 3.** The $k$-coalitional Shapley value can be calculated from Partition Decision Trees in the polynomial time.

**Proof.** Each path of the Partition Decision Trees can be translated into a set of Embedded MC-Nets rules (see Figure 2 for an example). In such rules, boolean expressions cannot be merged and the graph of restrictions is a clique. As for cliques, the chromatic

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1 For $k = 1$ there exists (only one) $k$-coloring if graph have no edges. For $k = 2$, each connected component has two $k$-colorings if it is bipartite (otherwise, zero $k$-colorings exists for the whole graph).
polynomial (and the number of colorings) is known, so using Theorem 2 we get a polynomial algorithm.

5 Conclusions

In this paper, we introduced \( k \)-coalitional cooperative games – a new class of cooperative games especially designed to study settings, where no more than \( k \) coalitions can simultaneously co-exist. We proposed a natural extension of the Shapley value to these games and studied its computational properties. In future work, we plan to consider an application of \( k \)-coalitional cooperative games to construct a model of epidemics diffusion in a social network. Here, each of \( k \) coalitions corresponds to one epidemic, and the value of a coalition could be defined as the number of nodes it has a maximal influence on (where influence is measured as a number of neighbours infected by this disease).

References


