Core-stable networks with widespread externalities

László Á. Kóczy

1 Centre for Economic and Regional Studies, Hungarian Academy of Sciences
Budaörsi út 45., H-1112 Budapest
koczy@krtk.mta.hu

2 Keleti Faculty of Business and Management, Óbuda University

Abstract. We introduce a core-based stability concept for networks with widespread externalities. The model is a generalisation of the recursive core for partition function form games. We present a simple example of a favour network and show that the core is nonempty when players must pay transfers to intermediaries. This simple setting also models economic situations such as airline networks.

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1 Introduction

Networks are used to describe a wide class of social and economic situations. Since Jackson and Wolinsky (1996) we are especially interested in the emergence of stable networks, networks that persist. In this seminal paper stability is driven by the persistence of links: while any player can refuse to maintain a link and thus any player can sever a link, only pairs of players can build new links. This leads to the natural concept of pairwise stability. This concept defines a wide class of stable networks. While these networks are immune to changes by single players or pairs, pairwise stability does not check for possible deviations by larger groups. Dutta and Mutuswami (1997); Jackson and van den Nouweland (2005) consider strong Nash equilibria (Aumann, 1959), where players are permitted to make coordinated, but noncooperative deviations. Such stable networks are very difficult to find, indeed strong Nash equilibria are rather rare and so several alternative models have been proposed. In the noncooperative setting one must very clearly specify the players’ strategies and their abilities to change them. For instance Calvó-Armengol and Ilkılıç (2008) consider the relation of equilibria under single link building and multilink severance. A cooperative approach allows to step over this issue. Ju (2013) considers coalitional cooperation in networks, but there the networks are exogenous and the means of cooperation are somewhat more limited. Ours is a general, cooperative approach.

Another problem is the seemingly inherent conflict between stability and efficiency. Already Jackson and Wolinsky (1996) find this conflict: unless we
insist on such an extreme allocation rule as the egalitarian there does not, in general, exist a network that is both stable and efficient. The conflict is due to the conflict between the parties forming the link on the one hand and the externalities generated by those links to the rest of the network. Jackson and Wolinsky (1996) assume that there are costs to forming links, but there are widespread benefits of being better connected. Morrill (2010) assumes that a new relationship is beneficial to the parties, but that the new connection may harm the rest of the network. The case of an employment network is cited, where acquaintances help players to new jobs, but the value of a connection depends on its exclusivity; the co-author network of Jackson and Wolinsky (1996) is another perfect example. Möhlmeier et al. (2013) combine the connection model with the co-author model allowing for positive externalities of link formation via the connections it provides, while increasing the congestion at the end nodes creating negative externalities at the neighbours. Buechel and Hellmann (2012) show that positive externalities may lead to under-connected, while negative externalities to over-connected stable networks (with both terms formally defined). We take a different, purely cooperative approach, where coalitional improvements are possible: a coalition can freely rearrange its internal structure of connections and may also jointly optimise the connections to other players. As a special case, the grand coalition may also act as a deviating coalition and so that stability naturally implies efficiency.

While coalitional deviations are also accounted for under strong stability, there is a fundamental difference between the noncooperative and cooperative approaches. Under strong stability coalitional deviations are unilateral and simultaneous. Our cooperative approach is based on the core. The core is a static concept: when a core allocation is proposed, it is accepted by all without protest. On the other hand, deviations from a non-core proposal are modelled in a dynamic way. If the proposal on the table does not meet the expectations or demands of one of the coalitions, it makes a threat of leaving the full cooperation if its demands cannot be met. If the threat works, the original proposal is abandoned, and a new one is made. If the threat does not work, the coalition in question leaves the joint agreement and begins to act according to its own interests. Due to the externalities its value will depend on the network structure the remaining players form.

The structure of the paper is then the following. First we introduce the general notation, introduce the game form we use to model the coalitional network games and recall the recursive core that inspires the solution concept that is introduced in the next section. We study two simple networks at length, determine the stable structures and discuss their possible applications.

## 2 Preliminaries

We consider a network game, where the payoffs of players or groups of players depend on the network, that is, the underlying graph. Csercsik and Kóczy (2011) describe a partition function form game of generators and consumers over an
electric power network, but ignore the possibility of building or severing a new edge, that is, a new powerline. Here we generalise that setting by allowing players or groups of players to build or severe links. The coalitional payoffs depend explicitly on the graph structure. Our notion of coalition stability is based on the stability notion of the (coalition structure) core. When a coalition deviates from the current structure it can (i) remove any link originating from one of its members, (ii) build a feasible link between two of its members (iii) allocate the value of the coalition among its members. Effectively the coalition can arbitrarily change its internal links, but external links can only be removed. The value of the coalition can only be determined once the graph structure is known. For this we must determine the structure of the remaining, residual players.

There are several possible approaches to take. By considering strong Nash equilibria Dutta and Mutuswami (1997) essentially assume that the structure remains unchanged. This is, however, not very likely given the widespread externalities present in the network: the deviation will change the payoff of the remaining players, who will react to the externalities in a complex way: they might form coalitions who also optimise their internal structure as well as choose the outgoing links they want to have or keep. The value of the deviating coalition will, in turn, also depend on the reaction chosen by the residual players so it is natural to take a conservative approach and expect the worst: if the the deviation is profitable in the worst case, it is surely profitable.

While this approach is very conservative from the viewpoint of the deviating players, the resulting set of coalitionally stable networks may include networks that only appear stable because of this assumption of extreme pessimism. In other words, extreme pessimism corresponds to extreme optimism regarding stability. More importantly, this approach ignores the interests of the residual players. So, along the lines of the recursive core (Kóczy, 2007) we assume that the residual players play a similar game and pessimism is applied only to the resulting, often unique coalitionally stable network or rather the payoffs induced by these residual structures.

We do not assume that the deviating coalition breaks all ties with the rest. The residual problem may therefore contain some outgoing links: these links can be broken, but no new outgoing links can be built. The latter follows from the fact that this would require mutual consent, that is, endpoints belonging to the same coalition. The outgoing links will be formally connecting to an artificial, non-strategic player 0 representing the outside world; when a set of players leaves the game, the links to this set are simply remapped to end at 0. A player may have links to more than one of the deviating players and these links may have very different implications therefore we keep all of these. As a result there may be multiple links to the outside world requiring us to use somewhat more general notation. This more general notation, on the other hand, permits us to consider rather general problems. For instance, in the case of a power network the different parallel arcs may be power lines with different transmission capacities and solving this game may determine not only where the lines should be built, but also what
their capacities should be. In the case of an existing line an upgrade could be modelled likewise.

Now we move on to the formal model of this graph.

2.1 Formal model

Let \( N \) denote the finite set of players, \( 2^N \) the set of all subsets, called coalitions, and \( g^N \) the set of all subsets of size 2. For a set \( S \) let \( \bar{S} \) denote its complement \( N \setminus S \), \( \Pi \) denotes the set of partitions of \( N \) into non-overlapping subsets, partitions of \( S \subseteq N \) are denoted \( \Pi(S) \). Let \( G = \{ g \mid g \subseteq g^N \} \) denote the set of all possible graphs over \( N \). It is common to define the value of a network as \( v : G \rightarrow R \).

Such a network, however allows no multiple or open links.

Let \( N_0 = \{0\} \cup N \) denote the set of players including a non-strategic player representing the outside world. \( G_0 \) is defined similarly to \( G \). We then define the set of (feasible) links \( \ell \). There is a well-defined mapping \( e : L \rightarrow G_0 \), where \( e(l) \) is simply the set of endpoints of \( l \in L \), but \( e(k) = e(l) \) does not imply \( k = l \). In other words the links in \( L \) have names – to identify multilinks between two players. We call \((N, L)\) or \( L \) an open multigraph or shortly graph. In this setup nodes \( i \) and \( j \) are connected by a link if there exists \( l \in \ell \) such that \( e(l) = \{i, j\} \).

It will not lead to confusion if we just say that \( ij \in \ell \). We say that \( \pi = \{l_k\}_{k=1}^m \) is a path of length \( m \) and the distance between nodes \( i \) and \( j \), denoted \( s(i, j) \) is simply the length \( m \) of the shortest path with \( i \in e(l_1) \) and \( j \in e(l_m) \).

We define the value of a coalition of players in a network by the network function

\[
V : 2^N \times L \rightarrow R.
\]  

(1)

This function assigns a real value to each coalition in a network, a value \( V(C, \ell) \) to each coalition \( C \) given a network \( \ell \). The triple \((N, L, V)\) is a cooperative game in network form or simply a game. We call a pair \( \omega = (x, \ell) \) consisting of a payoff vector and \( \ell \in L \) an outcome if it satisfies the following:

\[
\sum_{i \in N} x_i = \sum_{i \in N} V_i(\ell),
\]

in other words transfers are permitted among players within coalitions.

Note that a network function that is constant over graphs that partition players into the same connected components is actually a partition function therefore the network function form generalises the partition function form (Thrall and Lucas, 1963) by also accounting for the internal structure of components.

Let us denote the set of outcomes in \((N, L, V)\) by \( \Omega(N, L, V) \). The aim of this paper is to find the outcomes that cannot be improved upon by any coalition of players including the grand coalition.

2.2 The recursive core for partition function form games

In a partition function form game whether a coalition benefits from deviating depends on the induced partition of the players. The \( \alpha \)-core (Aumann and Peleg,
1960) assumes that a coalition deviates only if it gets a higher payoff irrespective of the induced partition. The core stability (Shenoy, 1979) is more permissive: a coalition deviates if any of the induced partitions gives a higher payoff. In the \( \gamma \)-core (Chander and Tulkens, 1997) the coalition must face individually best responses. Here we recall the concept of the recursive core (Kóczy, 2007, 2009), that allows the remaining, residual players to freely react and form a core-stable partition before the payoff of the deviating coalition is evaluated. The endogenous coalition formation process described by Ray and Vohra (1999) is inspired by a similar recursive idea.

First we define the residual game over the set \( R \subseteq N \). Assume \( R = N \setminus R \) have formed \( P_R \in \Pi(R) \). Then the residual game \( (R, V_{P_R}) \) is the partition function form game over the player set \( R \) with the partition function given by \( V_{P_R}(C, P_R) = V(C, P_R \cup P^{-R}) \).

**Definition 1 (Recursive core).** (Kóczy, 2007) For a single-player game the recursive core is trivially defined. Now assume that the recursive core \( C(N, V) \) has been defined for all games with \( |N| < k \) players. We call a pair \( \omega = (x, P) \) consisting of a payoff vector and a partition \( P \in \Pi(N) \) an outcome. Let us denote the set of outcomes in \( (N, V) \) by \( \Omega(N, V) \). Then for an \( |N| \)-player game an outcome \( (x, P) \) is dominated if there exists a coalition \( Q \) forming partition \( P_Q \) and an outcome \( (y, P_Q \cup P^{-Q}) \in \Omega(N, V) \), such that \( y_Q > x_Q \) and if \( C(Q, V_{P_Q}) \neq \emptyset \) then \( (y_Q, P_Q) \in C(Q, V_{P_Q}) \). The recursive core \( C(N, V) \) of \( (N, V) \) is the set of undominated outcomes.

The recursive core is well-defined, though it may be empty.

### 3 The recursive core for games in network function form

Our new coalitional stability concept is introduced in this section. The concept is an adaption of the recursive core for the network setting where the payoff of players or coalitions depends not on the partition, but on the network structure of the players. This more general game requires some more complex notation and terminology. First we present this notation, as before, we define the way to derive a residual network game and finally present the recursive core for network games.

#### 3.1 Residual games

Now assume that a coalition \( R \) has left the game forming partition \( P_{\pi_R} \) and structure \( \ell_{\pi_R} \). In the following we define a game in network form that can be solved using the same methods as the original game.

The residual game is a triple \( (R, L_R, V_R) = (R, L^{R}_{\pi_R}, V^{R}_{\pi_R}) \) where \( R \) is simply the remaining set of players. In the following we define the restriction \( L^{R}_{\pi_R} \) of \( L \) to \( R \) and the network function \( V^{R}_{\pi_R} \).
Feasible links can be of two types: links among members of $R_0$:

$$L_R = \{ l | l \in L, e(l) \subseteq R_0 \}$$

(2)

plus links connecting $R$ and $\overline{R}$:

$$L_{R, \overline{R}} = \{ l | l \in L, e(l) \cap R \neq \emptyset, e(l) \cap \overline{R} \neq \emptyset \}.$$

(3)

Since $\overline{R}$ had deviated, cooperation, that is, building new links in addition to those in $\ell_{\overline{R}}$ is not possible between $R$ and $\overline{R}$. Therefore

$$L_{R, \overline{R}} = L_R \cup (L_{R, \overline{R}} \cap \ell_{\overline{R}}).$$

(4)

Since $\overline{R}$ is now in the outside world, for the links across the deviation we update the endpoint $\overline{R}$ by 0 in the function $e_R : L_{R, \overline{R}} \rightarrow g^{R_0}$ is given as

$$e_R(l) = \begin{cases} e(l) & \text{if } e(l) \subseteq R_0 \\ e(l) \setminus \overline{R} \cup \{0\} & \text{otherwise.} \end{cases}$$

(5)

In order to give the network function we must be able to re-merge the separated structures. A re-merged structure consists of all the links that the two structures contain with the exception of those connecting links in $L_{R, \overline{R}}$ that have been severed on one side.

$$(\ell_{\overline{R}}, \ell_R) = \left( (\ell_{\overline{R}} \cup \ell_R) \setminus L_{R, \overline{R}} \right) \cup \ell_{R, \overline{R}}.$$  

(6)

Then we can define the network function for the residual game

$$V^\ell_{R, \overline{R}}(\ell_R) = V(\ell_R, \ell_{\overline{R}}).$$  

(7)

### 3.2 The recursive core

Now we can define the recursive core for network games. The definition is analogous to that for partition function form games and is therefore recursive. First the core is defined for a trivial, single player game. Assuming the definition for all, at most $k - 1$ player games, we extend the definition to $k$ player games.

**Definition 2 (Recursive core for network games).** For a single-player game the recursive core is trivially defined.

Now assume that the recursive core $C(N, L, V)$ has been defined for all games with $|N| < k$ players and consider an $|N|$-player game $(N, L, V)$.

We say that the outcome $(x, P, \ell)$ is dominated if there exists a coalition $Q$ forming partition $P_Q$ with network structure $\ell_Q$ and $y_Q$ such that for all outcomes $\left( y, P_Q \cup P_{\overline{Q}}, (\ell_Q, \ell_{\overline{Q}}) \right) \in \Omega(N, L, V)$ satisfying

$$-(y_Q, P_Q \cup P_{\overline{Q}}, (\ell_Q, \ell_{\overline{Q}})) \in C \left( Q, L^\ell_Q, V^\ell_Q \right) \text{ if this core is nonempty, or}$$
\[-(y_Q, P_Q, \ell_Q) \in \Omega \left( Q, L_{QQ}^{\ell_Q}, V_{QQ}^{\ell_Q} \right) \] otherwise

we have \( y_Q > x_Q \). The recursive core \( C(N, L, V) \) of \( (N, L, V) \) is the set of undominated outcomes.

In the terminology of Kóczy (2007) this is the pessimistic version, where pessimism is on the part of the deviating players. There is a corresponding version with optimistic players, where it is sufficient if the deviation is profitable for any (thus not all) of the residual (core) outcomes. It is easy to verify that the optimistic recursive core is weakly contained in the pessimistic recursive core.

### 3.3 Relation to other network formation models

In a seminal paper on network formation Jackson and Wolinsky (1996) introduced pairwise stability. A network is pairwise stable if (1) no unconnected pair benefits from linking up and (2) no player benefits from deleting an existing connection.

This model is a special case of our model in the sense that the equilibrium is driven by pairs: this corresponds to a game where only pairs have positive values. Larger coalitions are either worthless or create no greater demands than the pairs within implying a balancedness condition for each coalition. Pairwise stability, on the other hand, is a fundamentally noncooperative equilibrium concept as a slightly modified version is presented by Bloch and Jackson (2007) makes it even clearer. In such games there is no discussion about possible reactions as the decisions are made simultaneously. In our model players are more farsighted calculating with the reaction of other players. This aspect makes the two sets of solutions or equilibria mutually non-inclusive in general and applies to most of the network models (Bloch and Jackson, 2006).

### 4 Example: favour network

We consider a very simple example of a network, where link formation creates externalities and a wider cooperation can result in more efficient outcomes.

We take the example of a favour network consisting of individuals who maintain friendships at some cost \( c \). For simplicity we assume that a friendship is mutual, but the costs of maintaining the friendship are not necessarily shared equally: in the usual TU fashion we envisage a complex system of transfers of who buys which beer to maintain the network. Having many friends is great, but now we are interested in friends’ friends. When a friend’s friend is hiring and we want to apply for the job, the friend can put in a good word for us. The same would not work if we would make direct contact as praising ourselves is not so credible. Similarly, more distant relations may have too little information about us. In sum, the benefit of the network is the number of secondary friends a player has. In the following we formalise this rule, define the payoff function and determine the emerging equilibrium network structures.
Let $N_i = \{ j \mid j \in N, \exists \ell : e(\ell) = \{ i, j \} \}$ denote the neighbours of node $i$ and let $N_i^2 = \bigcup_{j \in N_i} N_j \setminus \{ i \}$ denote the secondary connections of $i$. Let $d_i = |N_i|$ denote the degree of node $i$. Similarly, let $d_i^2 = |N_i^2|$ denote the secondary degree of node $i$. Note that the payoff does not depend on the coalition structure. The payoff of coalition $C$ embedded in partition $\mathcal{P}$ given the network $\ell$ is

$$V_C(\ell) = \sum_{i \in C} \left( d_i^2 - \frac{d_i^C}{2} \right). \tag{8}$$

Note the absence of outside nodes.

We would like to know what is the core of this game.

**Proposition 1.** The core of the favour network game may only contain efficient networks.

**Proof.** Since the payoff function does not depend on the partition of the players, the game is cohesive, that is, the grand coalition can achieve any configuration. If the network is not efficient, a deviation by the grand coalition can strictly improve it and can strictly increase the payoff of each of the players.

In the following we determine the efficient network structure. We discuss three main cases:

**Tree** If the underlying network $\ell$ is a tree we show that it must be a star. Assume that this is not the case and that $i \in \arg \max_j d_j$. Moreover let $k$ be a leaf not connected to $i$, but to $j$. Since $k$ is a leaf, $d_k^2 = d_j - 1$. Now modify $\ell$ such that the link between $j$ and $k$ is moved to $i$ and $k$. Since $i$ is still a leaf we get $d_k^2 = d_i^2 - 1 = d_i > d_j - 1 = d_k^2$, where the $d^2$-s refer to values in the modified network. Since no other indirect connections are affected, the net gain is positive. Therefore all nodes must be connected to the node with the highest degree resulting in a star.

**Graph with triangles** Now consider the case when the network is not a tree. Firstly assume that the underlying graph $\ell$ contains triangles. Consider a triangle $T = \{ f, g, h \}$, such that, without loss of generality $h \in \arg \max_{i \in T} d_i$. Observe that the value of the network increases if we move the links (except from those from $f$ and $h$) pointing to $g$ to $h$ instead. To be more precise: it can be shown that if there are such $i$ that are not connected to $h$ then the value of the network can be increased.

In a similar fashion we can move links to $f$ to $h$, too. As a result triangles are connected to the rest of the network via one of their vertices only.

**Larger cycles** Now we show that larger cycles cannot be part of an efficient network. For the moment assume that there are larger cycles, too. Due to the previous result, the cycle may only share vertices and not arcs with triangles. Consider the smallest cycle of length at least 4, $C$ – this has at least four nodes: let $h, i, j$ and $k$ nodes following each other on the cycle and let $h \in \arg \max_{m \in C} d_m$ be
one of the points with the highest degree in the cycle. By the result that \( \{h, i\} \) is not part of a triangle, \( N_h \) and \( N_i \) are disjoint. Then consider the following modification to the network: move the arc linking \( j \) and \( k \) to link \( j \) and \( h \). After the change \( j \) has \( d_h + d_i \) secondary neighbours, while before the change at most\(^3\) \( d_k + d_i < d_h + d_i \). Therefore if the graph has larger cycles, it can be made more efficient by creating a triangle and thereby breaking the cycle. A repetition of this step eliminates all cycles of length 4 or more.

After the elimination of large cycles, and following the recommended improvements, we get a graph, which looks a bit like a tree, but with some triangles attached to some vertices. Thanks to this similarity, we can improve this graph similarly to the improvement applied for trees:

Select \( i \in \arg \max_j d_j \). With more than 2 players and a connected graph we either do not have triangles or \( d_i > 2 \) in which case \( i \) cannot be one of the non-connecting vertices (the \( f \)'s and the \( g \)'s) of a triangle. Let \( f \) and \( g \) such non-connecting vertices of a triangle \( T = \{f, g, h\} \). Now modify the graph so that \( fh \) and \( gh \) are moved to \( fi, gi \). As before, by moving to a node with a higher degree, both \( f \) and \( g \) have more secondary connections. While the direct connections \( N_h \{f, g\} \) of \( h \) lose them, those in \( N_i \) gain them and by assumption \( d_i \geq d_h \).

Once we are done with the triangles, we have a node with many triangles attached to it, but for the rest, the graph is just like a tree. So let \( k \) be a leaf not connected to \( i \), but to \( j \). Since \( k \) is a leaf, \( d_k^2 = d_j - 1 \). Now modify \( \ell \) such that the link between \( j \) and \( k \) is moved to \( i \) and \( k \). Since \( i \) is still a leaf we get \( d_k'^2 = d_k^2 - 1 = d_i > d_j - 1 = d_k^2 \), where the \( d \)'s refer to values in the modified network. Since no other indirect connections are affected, the net gain is positive. Therefore all nodes must be connected to the node with the highest degree.

So far we have only looked at improvements that did not affect the number of direct connections, we merely rearranged them to have a more efficient structure. As a result we have a player at the centre and all other \( n - 1 \) players are linked to it. Some of these outer players \( f, g \) are directly connected. Such connections are never needed to have each other as secondary connections as this works via the central player. Outer links are used to have the central player as a secondary connection. The added value of such a link is therefore \( 4 - c \) if neither \( f \) nor \( g \) is connected to other non-central players, the value is \( 2 - c \) if one of them is connected and \( -c \) if both. For high \( c \) these links are severed, for low values of \( c \) non-central players link up in pairs, and if \( n \) is even (so that \( n - 1 \) is odd) the remaining non-central player links to another only if \( c < 2 \). The links to the centre only break if \( c > 2(n - 2) \).

Therefore if \( c > 2(n - 2) \) we get an empty network, for \( 2(n - 2) > c > 4 \) we get a star, for \( 4 > c \) we get a flower: if \( n \) is even, for \( c > 2 \) it has a stem, otherwise a double petal.

**Proposition 2.** The core of the favour network game is empty.

\(^3\) Note that \( i \) and \( k \) may have common neighbours.
Proof. The next question is stability. Consider a deviation by a single player forming a singleton coalition. If this player forms or keeps no links, it has a zero payoff. Let us see if it can have a higher payoff. Suppose it keeps a link with its highest-degree neighbour. Since the residual game will be similar to the original one, the players form a star or a flower. If so, it is always better to form it “around” the player with the external link. Thereby the deviating player becomes a peripheral player in a star with a payoff $n - 2 - \frac{c}{2}$. The total value of a star is $(n - 1)(n - 2) - (n - 1)c$. Since the star is formed by $n$ players, there is a player with a payoff of at most $(n - 1)(n - 2) - (n - 1)c/n$, therefore the deviation is profitable if

$$n - 2 - \frac{c}{2} > \frac{(n - 1)(n - 2) - (n - 1)c}{n}$$

This is satisfied when $c > 2(n - 2)$, but we have not tested the stability of the residual core. If it empty, the deviating player must expect the worst of all possible reactions, including the one where links to it are broken and therefore his payoff is 0. To check this, consider a more general case with $k$ players deviating. It is easy to see that these players will all be peripheral players who do not want to change the underlying network, only the distribution of the payoffs, so that all these players will keep their links to the central player and then efficient and therefore only possible reaction in the residual game is a star around that player.
The question is: will this player keep the links to the deviated players. What causes the problems? While the total value of the network does not change, the central player, by maintaining the external links, subsidizes the deviated players more and more. As the number of departed players increases the residual players’ benefit per link to the deviating players decreases, while the associated costs remain the same. The links remain profitable only if

\[ n - k - 1 > \frac{c}{2} \]

where \(0 < k < n\). For some \(k\) this will be violated and then the deviations are not profitable any more. Consider a deviation by \(k - 1\) peripheral players: the residual core is nonempty and the deviation will be profitable. Therefore the recursive core of this game is empty.

Note that this finding is driven by the fact that the central player must sacrifice himself to the benefit of others: Normally others compensate him for this, but selfish players may deviate and stop such transfers. In reality such a central player has a very strong position and gets rewarded for the favours he can provide. In the following example we make these rewards explicit by assuming that, upon forming a link between players \(i\) and \(j\), player \(i\) must pay a transfer to \(j\) that is proportional to \(d_j - 1\). As a result, a central player gets a high transfer, while a leaf gets nothing. Then the payoff of coalition \(C\) embedded in partition \(P\) given the network \(\ell\) is

\[
V_C(\ell) = \sum_{i \in C} \left( d_i^2 - \frac{d_i c}{2} + \left( d_i (d_i - 1) - \sum_{j \in N_i} (d_j - 1) \right) t \right),
\]

where \(t < 1\) is the compulsory transfer for using an intermediary.

**Proposition 3.** The core of the modified favour game is not empty if \(t\) is sufficiently high \(t > \frac{c+2}{2n}\).

**Proof.** Firstly observe that the modification merely introduces transfers among players, so that the value of the grand coalition does not change. In particular,
the efficient structures remain the same. We may therefore focus on the issue of stability. We limit our attention to star structures; the case when \( c < 4 \) is similar.

Consider a star, and consider a deviation by \( k \) peripheral players. What happens in the residual game? The former central player has already \( k \) connections to the deviating players. Due to our assumptions that no new links may form between coalitions, no other player can have external links. By linking to this player the remaining \( n - k - 1 \) players do not only get a very high payoff, but they also increase the value of this central player’s services to the deviating players. Formerly this was positive externality they could not benefit from, but now the deviators must pay a fee for it. So if the residual core is not empty, it keeps the pre-deviation structure. Is this core non-empty? To see this, first compare the payoffs of players in different positions (without the possible transfers within the coalition). We will show that the central player earns more. To see this, observe the following: What a player earns only depends on the network structure. The network structure has not changed due to the deviations. At last: the network, and the payoffs (recall we ignore transfers) are symmetric among the peripheral players. Therefore if we show that the average payoff is higher than the peripheral players’ payoff this shows the result.

A player on the periphery has a value \( n - 2 - (n-2)t - \frac{c}{2} \), while an average player has \( \frac{(n-1)(n-2)-(n-1)c}{n} \). We want to show

\[
\begin{align*}
    n - 2 - (n-2)t - \frac{c}{2} &< \frac{(n-1)(n-2)-(n-1)c}{n} \\
    \frac{c+2}{2n} &< t
\end{align*}
\]

That is, if \( t \) is sufficiently large, the central player earns more. In such a case the central player has no incentives to deviate and become a peripheral player, while a player can only become central by cooperation with all other players. This holds both in the original game and in the residual game, since the underlying networks are the same.

5 Remarks

We have introduced a rather general model to allow for coalitional improvements in a network with possible transfers among coalition members. We have used a very simple game, a favour network to illustrate this model. In the first version there was a tradeoff between direct costs and indirect benefits, while in a second model players with many friends got transfers in exchange of their services as intermediaries. While such transfers are quite natural even in the settings of such personal connections, our model fits rather well other situations. Consider the hub-and-spoke network of air travel. There are natural differences: direct connections are the most valuable, so let \( c \) denote the difference between the cost and the benefit of the connections. The problem is not very interesting if
the benefits exceed the costs - we will have a complete network. On the other hand we see an increasing cost of flights to major hubs. While technically it is not airports paying transfers to others, minor airports heavily subsidise flights to major destinations as it makes them attractive to travellers and the market power of airlines depends on the significance of airports they fly to (Borenstein, 1989). Do we see our model confirmed in real life? Yes, in the sense that our equilibrium network structure is a hub and spoke network, but our example did not allow for heterogeneity among the players, which might explain the emergence of particular cities as transportation hubs (Konishi, 2000). In real life popular hubs do not simply charge more, but due to congestion adding new flights might become prohibitively costly or even impossible due to capacity constraints, at lest in the short run. We are still working on the problem with explicit capacity constraints, but it is clear that the network will be less simple.

At last, in this model we have assumed that the arcs are not directed. The model and the results naturally generalise to open multi-digraphs, too.
Bibliography


